Scaling of the information length in 1D tight-binding models

1992 J. Phys.: Condens. Matter 4 149

(http://iopscience.iop.org/0953-8984/4/1/024)

J. Phys.: Condens. Matter 4 (1992) 149–156. Printed in the UK

Scaling of the information length in 1D tight-binding models

Giulio Casati†‡, Italo Guarneri§||, Felix Izrailev†§¶, Shmuel Fishman†† and Luca Molinari†‡

† Dipartimento di Fisica dell Universita di Milano, 20133 Milano, Italy
‡ Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Italy
§ Dipartimento di Fisica Nucleare e Teorica dell Universita di Pavia, 27100 Pavia, Italy
|| Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, Italy
¶ Institute of Nuclear Physics, 630090 Novosibirsk, USSR
†† Department of Physics, Technion–Israel Institute of Technology, Haifa 32000, Israel

Received 1 July 1991

Abstract. The localization of eigenfunctions in finite samples of the 1D Anderson and Lloyd models is quantitatively described by the information length. This quantity is numerically investigated for both models and it is found to scale with the size of the sample and the disorder according to a simple law.

1. Introduction

The dynamical phenomenon of localization, common to many models of quantum chaos, has often been compared with that of Anderson localization, occurring in disordered crystals [1]. The diffusive absorption of energy that follows the appearance of chaos in periodically perturbed nonlinear classical systems is strongly limited and may even be suppressed by quantization. The 'kicked rotator' has been a prototype for this phenomenon and in that case a formal connection with models of the Anderson type has been found [2]. In both cases one has to deal with eigenfunctions—of the Hamiltonian in the Anderson case, of a Floquet operator in the dynamical case—that are to some extent localized inside a finite 'sample' of a fixed size. These eigenfunctions have a random aspect and their statistical properties may be described in the language of multifractals [3]. How various statistics related to such eigenfunctions scale with the size of the sample and with the disorder is also a common problem in the two cases. In the dynamical case this issue has been investigated by analysing the behaviour of the so called 'entropic localization length' of eigenfunctions [4, 5], which was indeed found to obey a scaling law. The band structure of the relevant Floquet matrix and the high degree of randomness of the matrix elements then suggested that such a scaling could be more generally a property of band-random matrices, which indeed was shown to be true [6].

In the case of the Anderson model the scaling properties are not usually referred to the entropic length but to the inverse Lyapunov exponent, which can be efficiently computed by means of the transfer matrix method and which is moreover directly related to the residual conductance via the Landauer formula [7]. Therefore, a direct comparison of these different scaling laws is not possible. On the other hand, the

0953-8984/92/010149+08$03.50

entropic length (and, possibly other generalized lengths, like the inverse participation ratio) appear to be much more convenient when a very large number of sites are coupled by the interaction. This is typical in the semi-classical regime of models that are endowed with a well-defined classical limit and in such cases a transfer matrix approach is impossible. In order to compare the scaling properties of dynamical localization and of band-random matrices to those of Anderson localization, one has therefore to reformulate the latter in terms of entropic or related lengths.

In this paper we investigate the scaling properties of the entropic length for the Anderson and for the Lloyd model. Our numerical results yield evidence that these models exhibit a similar scaling law. Though it is not clear how this scaling law may be related to the scaling theory of conductance, it has the important property of taking the same form in both cases, as soon as a proper choice of variables is made. This raises the interesting question of whether the same scaling law can be found in different models of localization.

2. The tight-binding Hamiltonian

The tight-binding models we investigate are characterized by Hamiltonians with a tridiagonal symmetric structure and random diagonal entries. The eigenvalue equation is

$$(Hu)_n = u_{n+1} + V_n u_n + u_{n-1} = E u_n \tag{1}$$

The boundary condition are $u_0 = u_{N+1} = 0$ and the potential $\{V_n\}$ is a set of N independent random variables, with the same probability distribution $P(V)$.

We shall consider the two important cases of the Anderson and Lloyd models, with probabilities given respectively by:

$$P_W(V) = \begin{cases} 1/W & \text{for } -W/2 < V < W/2 \\ 0 & \text{elsewhere} \end{cases} \tag{2}$$

$$P_W(V) = \frac{1}{\pi} \frac{W}{V^2 + W^2} \qquad (W > 0) \tag{3}$$

Other models which have largely been studied in this context are characterized by a non-random potential, specified by a periodic function whose period is incommensurate with the lattice spacing. Two examples are Harper's (or Mathieu) equation [8] and the 'Maryland model' [9].

It is mathematically proven that the above random models in the large N limit display exponentially localized eigenfunctions, no matter how small the disorder; the rate of decay is measured by the Lyapounov exponent γ which may be evaluated by Thouless' formula [10] or by the transfer matrix method [11]. The latter is a convenient method for translating the recurrence relation for the eigenfunction into a multiplicative procedure

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \left[\prod_{k=1}^{n} \begin{pmatrix} E - V_k & -1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} \tag{4}$$

Since $u_0 = 0$, with $u_1 = 1$ this is precisely the recurrency property for the determinants of the Jacobi submatrices of H, and therefore $u_{N+1} = \mathrm{Det}(E - H)$, which shows that the eigenvalues are determined by the boundary condition $u_{N+1} = 0$. Although γ for a finite N depends on the realization of the disorder, in the limit $N \to \infty$ it converges to a non-random value, the inverse of which is known as the *localization length* ξ_∞. In the Anderson case, the dependence of the localization length on energy is rather complicated. For small disorder (small W) it has been evaluated in [12, 13] for different energy ranges. For example, at $E = 1$ it was found that

$$\xi_\infty^{-1} = (W^2/72)[1 - (2/3)W^{2/3} + (1/60)W^2 + \cdots]. \tag{5a}$$

In the opposite case of large disorder ($E \ll W$) the behaviour is

$$\xi_\infty^{-1} = \log(W/2) - 1 + \cdots. \tag{5b}$$

For the intermediate range one has to compute numerically the rate of exponential growth of vectors under repeated application of the transfer matrices for very long samples (figure 1).

As for the Lloyd model, the Lyapunov exponent can be found analytically in the limit $N \to \infty$:

$$\gamma = \cosh^{-1}\left(\frac{1}{4}\sqrt{(2+E)^2 + W^2} + \frac{1}{4}\sqrt{(2-E)^2 + W^2}\right). \tag{6}$$

The scaling theory for the metal–insulator transition has received an elegant formulation in the formalism of tight-binding models, where it relates the inverse Lyapunov exponent ξ_N for finite samples of size N to the *localization ratio* ξ_∞/N. The scaling assumption is [7]

$$\frac{\xi_N}{\xi_\infty} = f\left(\frac{\xi_\infty}{N}\right) \tag{7}$$

where $f(x)$ is a scaling function. The form of this function is very important, because it is directly related to the behaviour of the conductance as a function of the sample size [7].

In this paper we characterize the structure of eigenfunctions by means of a different parameter: the *information*, or *entropic localization length* [4], which is defined in terms of the Shannon entropy, as follows. The Shannon entropy of a normalized state $(u_1, u_2, \ldots, u_N)$ is:

$$H[u_1, \ldots, u_N] = -\sum_{i=1}^{N} u_i^2 \log u_i^2. \tag{8}$$

In our study we consider ensembles of states specified by the value of the energy E and by different realizations of the random potential. For such ensembles, we define the average normalized information length:

$$\beta(E, N, W) = \exp[\overline{H} - H_{\mathrm{ref}}] \tag{9}$$

where $\overline{H}$ is the entropy of the state of energy E, averaged over disorder, and H_{ref} is a normalization entropy computed as the average entropy in some reference ensemble.

The great advantage of this definition is the applicability in both extended and localized states; moreover, it has been shown to correspond to the common intuition of the fraction of unperturbed states which, on the average, are significantly populated by eigenstates with the given energy [7, 14–16].

In our case, we choose as reference ensemble, which by definition will have $\beta = 1$, that which corresponds to maximally delocalized states. These are obtained in the limit of vanishing disorder and have the form of plane waves. The eigenvalues are $E^{(k)} = 2\cos[k\pi/(N+1)]$, $k = 1, \ldots, N$ with eigenfunctions

$$u_n^{(k)} = \sqrt{\frac{2}{N+1}} \sin\left(n\frac{k\pi}{N+1}\right) \qquad n = 1, \ldots, N. \tag{10}$$

These eigenfunctions represent the limiting case of infinite localization ratio; no randomness survives in them. Their entropy for large N has the same value irrespective of the label k of the eigenvalue: $H_{\text{ref}} = \log(2N) - 1$. This is an important difference to the case of band-random matrices, where the maximally extended states are completely random ones. In that case, the reference ensemble is the microcanonical ensemble and the reference entropy has to be computed accordingly [4, 14].

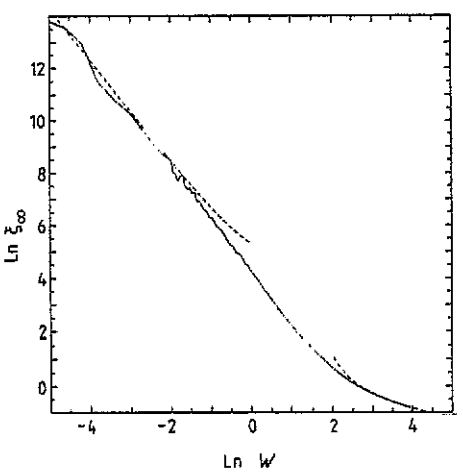

Figure 1. Numerically computed localization length ξ_∞ versus disorder for the Anderson model for $E = 1$. The broken curves represent respectively the theoretical limits at low disorder (equation (5*a*)) and high disorder (equation (5*b*)). The experimental line is obtained by averaging over disorder and for a sample of length 2×10^6 for $\ln(W) < -2$ (where fluctuations are larger) and length 10^5 for $\ln(W) > -2$.

3. Numerical results

The numerical work reported here was aimed at investigating how the information length (9) scales with disorder and with sample size. The scaling law (7) suggests that $\beta(N, E, W)$ (which for any given N, E, W yields a measure of the spread of the eigenvectors) may be essentially determined by the sample size N and by the localization ratio ξ_∞/N. As a matter of fact, this sort of scaling has been actually found to hold for

the Kicked Rotator model [5] and for band-random matrices [6]. Our present results for the Anderson model (figure 2) provide evidence of a scaling law of the form:

$$\beta = \frac{e^C \xi_\infty N^{-1}}{1 + e^C \xi_\infty N^{-1}}. \tag{11}$$

The same scaling is also made apparent by the log–log plot in figure 3, which demonstrates the following dependence equivalent to (11):

$$\log \frac{\beta}{1-\beta} = \log\left(\frac{\xi_\infty}{N}\right) + C \tag{12}$$

In our computations on the Anderson model, E was taken in two different 'energy windows' of width $\Delta E = 0.1$, centred at $E = 0.1$ and at $E = 1.0$ respectively. The value of β was obtained by statistically averaging over an ensemble of random samples of size $N = 400$–3200. For every fixed sample size, a large number of results (up to 1000) were generated. For every sample we computed the average entropy of the eigenstates whose energies were found to lie in the chosen window; the result was further averaged over the different samples of the same size and finally substituted into (9).

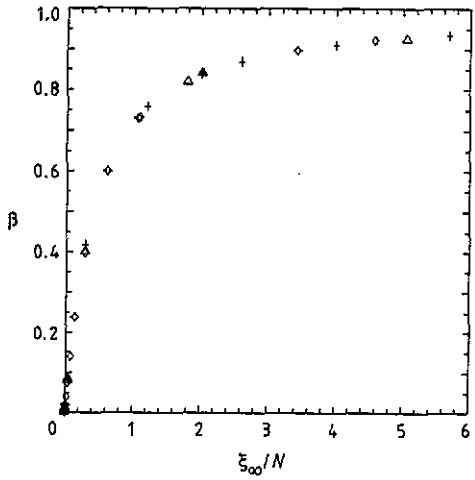

Figure 2. Scaling of β versus the localization ratio $x = \xi_\infty/N$ for the Anderson model for $0.05 < E < 0.15$ and $N = 3200$ (+), $N = 1600$ (○), $N = 800$ (△) and $N = 400$ (+). The same symbols are used in the following figures.

Since no analytical expression of ξ_∞ is available for the whole range of disorder covered by our computations, we resorted to a numerical computation of ξ_∞ via the transfer matrix method, except for the case of very weak localization, where the theoretical weak-disorder expansion (5*a*) was used. A plot of the numerically computed ξ_∞ versus W at $E = 1.0$ is shown in figure 1, together with the theoretical results (5*a*), (5*b*) for the cases of weak and strong disorder respectively.

For the constant C in (12) our data yield $C \approx 1$ for both the windows at $E = 0.1$ and $E = 1.0$. It is interesting to compare these numerical values with an approximate estimate of C that can be obtained as follows. In the limit of very small

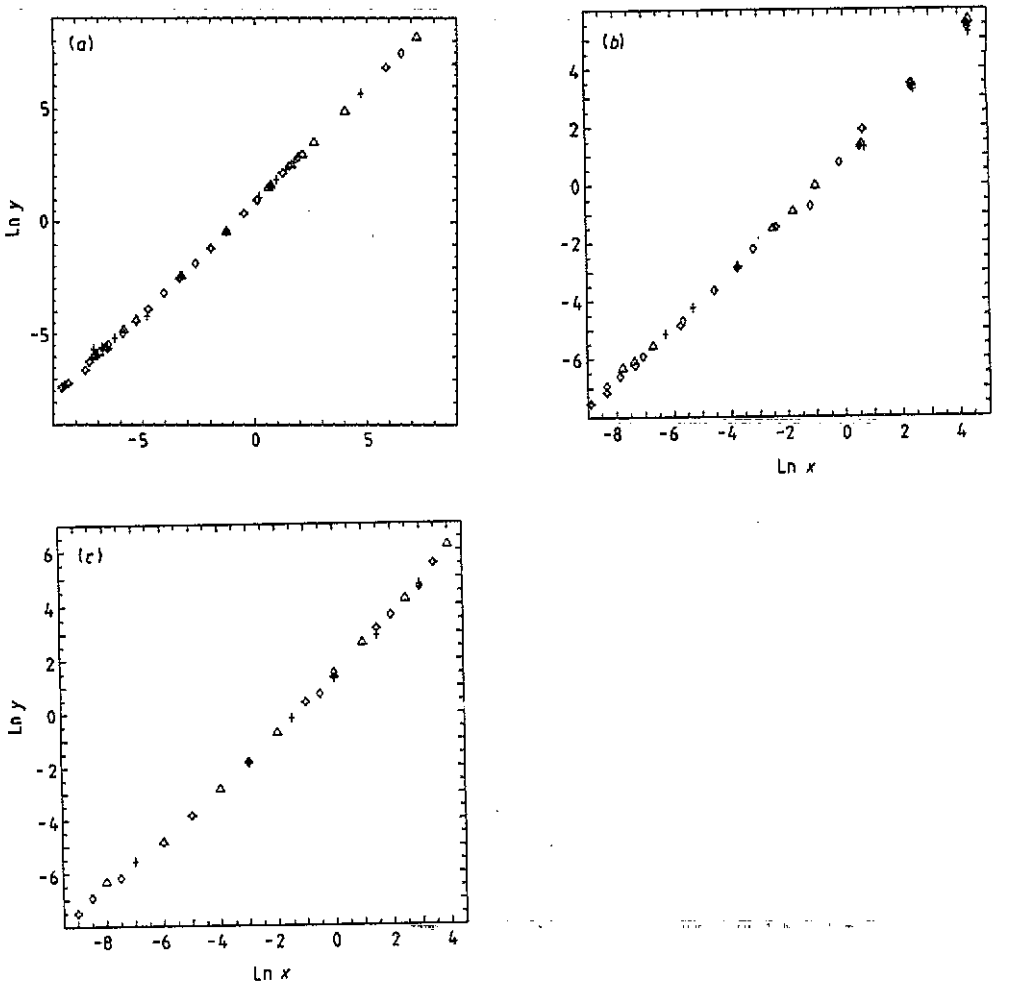

Figure 3. The scaling of β versus the localization ratio $x = \xi_\infty/N$ in the variables $\ln y = \ln(\beta/1-\beta)$ and $\ln(x)$. (*a*) Anderson model, with energy window $0.05 < E < 0.15$. (*b*) Anderson model, with energy window $0.95 < E < 1.05$. (*c*) Lloyd model, with energy window $0.95 < E < 1.05$.

localization ratio one can assume the eigenvectors to decay exponentially away from a single peak, with the average shape $u_n \approx \exp(-\xi_\infty^{-1}|n-n_0|)$. In the limit of large N, this average dependence can be used to compute the average entropy of eigenfunctions corresponding to a given energy and then the corresponding β; in this way one finds $\beta = (e^2/2)(\xi_\infty/N)$. Upon substituting this result in equation (9) one gets $C = 1.307\ldots$. The discrepancy with the actual value of C found from numerical data is due to the fact that the assumed average exponential form of the eigenfunction does not account for unknown fluctuations that change the numerical factor in the expression of β.

Similar computations were made on the Lloyd model. Also in this case a scaling law of the form (12) was found. Figure 3(*c*) shows results for an energy window centred at $E = 1.0$. For matrices of rank 3200 the disorder parameter W ranges from $W = 12$ to $W = 7 \times 10^{-6}$, so that the scaling actually holds in a very large interval.

In this case the data give $C \approx 1.4$.

4. A model-independent form of the scaling law

Because the value of the constant C in the scaling law (12) depends on the model (Anderson or Lloyd) and on the energy window as well, the scaling law as expressed in (12) is not the same in all those cases. However, the fact that this law is *linear* in all cases allows for an interesting invariant reformulation.

In all cases we found a scaling relation that, upon substituting in (11) the definition (9) of β, takes the general form:

$$\frac{e^{\overline{H}(N,W)}e^{-H_{\rm ref}(N)}}{1-e^{\overline{H}(N,W)}e^{-H_{\rm ref}(N)}} = K\frac{\phi(W)}{N} \tag{13}$$

where W is a measure of disorder, $\phi(W)$ is an 'intensive' (i.e. N-independent) parameter like ξ_∞ for Anderson and Lloyd or W^2 for band-random matrices (in that case W is the bandwidth), and finally N is the size of the system. The energy dependence is here unimportant and we have dropped it. The value of K depends on the model and on the energy. In any case, one has the large-N asymptotics $H_{\rm ref}(N) \sim \log(N/c)$ (with c being a model-dependent constant) or $\exp[-H_{\rm ref}(N)] \sim c/N$. Upon substituting this in (13) and taking the limit $N \to \infty$ one finds

$$ce^{\overline{H}(\infty,W)} = K\phi(W)$$

that can be used to eliminate $K\phi$ from (13); in this way the constant c also drops out and one finally finds

$$e^{-\overline{H}(N,W)} = e^{-\overline{H}(\infty,W)} + e^{-\overline{H}(N,0)} \tag{14}$$

The sample size N now enters through $\exp[H_{\rm ref}(N)]$ which may be thought of as an effective size of the sample. By defining the length $d(N,W) = \exp\overline{H}(N,W)$ for the sample of N sites and disorder parameter W, we therefore obtain:

$$\frac{1}{d(N,W)} = \frac{1}{d(\infty,W)} + \frac{1}{d(N,0)}\,. \tag{15}$$

5. Conclusions

The phenomenon of quantum localization is common to several models, including some that are not directly related to electronic transport in disordered solids and are quite different from the tight-binding models where that phenomenon was first identified. This is the case of models of 'quantum chaos', where important progress was made possible by the use of ideas and concepts from the theory of the Anderson localization. In particular, a scaling property was found in the Kicked Rotator model for the 'information length' of localized eigenstates. In order to assess precisely to what extent this sort of scaling is similar to the scaling in 1D tight-binding models, we set out to investigate the scaling properties, if any, of the same quantity in the Anderson and the Lloyd model. We have found numerical evidence that these models also display a scaling law for the information length; moreover, this scaling law assumes a universal form as soon as it is formulated in terms of 'informational' quantities. The simple and elegant form of this law (15) calls for a theoretical explanation and also raises a question, as to whether it can be expected to hold in any 1D localization problem.

Acknowledgments

This work has been supported by a project of CNR for parallel calculus, and partly by the US–Israel Binational Science Foundation (BSF). We want to thank Spiros Evangelou for useful discussion.

References

[1] Fishman S, Grempel D R and Prange R E 1982 *Phys. Rev. Lett.* **49** 509
[2] Fishman S, Prange R E and Griniasty M 1989 *Phys. Rev.* A **39** 1628
[3] Evangelou S N 1990 *J. Phys. A: Math. Gen.* **23** L317
[4] Izrailev F M 1990 *Phys. Rep.* **196** 299
[5] Casati G, Guarneri I, Izrailev F and Scharf R 1990 *Phys. Rev. Lett.* **64** 5
[6] Casati G, Molinari L and Izrailev F 1990 *Phys. Rev. Lett.* **64** 1851
[7] Pichard J L 1986 *J. Phys. C: Solid State Phys.* **19** 1519
[8] Thouless D J 1990 *Commun. Math. Phys.* **127** 187
[9] Grempel D R, Fishman S and Prange R E 1982 *Phys. Rev. Lett.* **49** 833
[10] Thouless D J 1972 *J. Phys. C: Solid State Phys.* **5** 77
[11] Schmidt H 1957 *Phys. Rev.* **105** 425
[12] Derrida B and Gardner E 1984 *J. Physique* **45** 1283
[13] Campanino M and Klein A 1990 *Commun. Math. Phys.* **130** 441
[14] Izrailev F M 1988 *Phys. Lett.* **134A** 13
[15] Reichl J 1988 *Europhys. Lett.* **6** 669
[16] Blumel R and Smilansky U 1984 *Phys. Rev.* A **30** 1040